Simplest Generalization of Pell's Problem

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Introduction. "Pell's Problem" is to display integer solutions of the Diophantine equation

$$x^2 - ny^2 = 1 \quad : \quad n \text{ not a perfect square}$$

The problem was already ancient by the time it was posed by John Pell [1611–1685] and—a few years before him—by Fermat. The case $n = 2$ had been considered by Pythagorous and contemporary Indian mathematicians; it had been discussed in some generality by Brahmagupta [598–c.670] and the general solution was provided by Bhaskara in 1150. Its mistaken attribution to Pell is due to Euler, after whom the problem attracted the interest of (among others) Lagrange.[1]

The material reported here derives from recent correspondence with Ahmed Sebbar, of the Institut de Mathématique de Bordeaux, Université de Bourdeaux, with whom I have recently been in correspondence in mainly other connections.

Algebraic group implicit in Pell's equation. The requirement that the matrix

$$\mathbb{M} = \begin{pmatrix} x & \sqrt{n}y \\ \sqrt{n}y & x \end{pmatrix} \tag{1}$$

be unimodular ($\det \mathbb{M} = 1$) entails

$$x^2 - ny^2 = 1 \tag{2}$$

which inscribes a hyperbola ("Pell-Fermat conic") on the xy-plane. The set of such matrices is multiplicatively closed

$$\begin{pmatrix} x_1 & \sqrt{n}y_1 \\ \sqrt{n}y_1 & x_1 \end{pmatrix}\begin{pmatrix} x_2 & \sqrt{n}y_2 \\ \sqrt{n}y_2 & x_2 \end{pmatrix} = \begin{pmatrix} x_{12} & \sqrt{n}y_{12} \\ \sqrt{n}y_{12} & x_{12} \end{pmatrix} \tag{3}$$

$$x_{12} = x_1x_2 + ny_1y_2$$
$$y_{12} = x_1y_2 + y_1x_2$$

[1] For further information and references concerning this rich subject, see the excellent Wikipedia article "Pell's Equation."

Associativity is automatic, commutativity is easily confirmed, and the set obviously contains the identity $\mathbb{I}$ among its elements. We can therefore look upon such matrices as representatives of the elements of a certain Abelian group. The group composition law can be notated

$$\{x_1, y_1\} * \{x_2, y_2\} = \{x_1x_2 + ny_1y_2, x_1y_2 + y_1x_2\} \tag{4.1}$$

The identity has in this notation become $\{1, 0\}$ and the inverse is given by

$$\{x, y\}_{\text{inverse}} = \{x, -y\} \tag{4.2}$$

Forming the determinants of the left/right sides of (3) we obtain Brahmagupta's identity

$$(x_1^2 - ny_1^2)(x_2^2 - ny_2^2) = (x_1x_2 + ny_1y_2)^2 - n(x_1y_2 + y_1x_2)^2 \tag{5}$$

from which it follows that

$$\text{if } (x, y)_n \text{ is a pair of Pell integers then so is } (x^2 + ny^2, 2xy)_n \tag{6.1}$$

and that

$$\begin{array}{l}\text{if } (x_1, y_1)_n,\ (x_2, y_2)_n \text{ are two such Pell pairs, then} \\ \text{so is } (x_1x_2 + ny_1y_2, x_1y_2 + y_1x_2)_n \text{ a Pell pair}\end{array} \tag{6.2}$$

For example, the leading Pell pair in the case $n = 2$ is $(3, 2)_2$, which by (6.1) yields the Pell pair $(17, 12)_2$. Introducing those pairs into (6.2) gives $(99, 70)_2$. We note in passing that the line $(3, 2)_2 \longrightarrow (17, 12)_2$ has slope $\frac{10}{14}$, while the line $(1, 0)_2 \longrightarrow (99, 70)_2$ has slope $\frac{70}{98}$, and that $\frac{10}{14} = \frac{70}{98}$. This reflects a remarkable general circumstance:

$$\frac{\sqrt{\frac{x_2^2-1}{n}} - \sqrt{\frac{x_1^2-1}{n}}}{x_2 - x_1} = \frac{x_1\sqrt{\frac{x_2^2-1}{n}} + x_2\sqrt{\frac{x_1^2-1}{n}}}{x_1x_2 + n\sqrt{\frac{x_1^2-1}{n}}\sqrt{\frac{x_2^2-1}{n}} - 1}$$

When one consults tables of leading Pell pairs $(x, y)_n$ for ascending values of n (such as the one that appears in the Wikipedia article cited above), two cases stand out. The first of those is the famous case

$$(1766319049, 226153980)_{61}$$

and the next—which I display in the context of its neighbors—is

$$\begin{aligned} &(1351, 130)_{108} \\ &(158070671986249, 15140424455100)_{109} \\ &(21, 2)_{110} \end{aligned}$$

Sebbar's generalization. Sebbar, noting that the matrix $\mathbb{M}$ is a circulant matrix, looks to the group of unimodular circulant matrices of next higher order

$$\mathbb{S} = \begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix} = x\mathbb{I} + y\mathbb{P}_1 + z\mathbb{P}_2 \tag{7}$$

The matrices $\mathbb{P}$ are permutation matrices, and provide a representation of the cyclic group of order 3: $\mathbb{P}_1^2 = \mathbb{P}_2, \mathbb{P}_1^3 = \mathbb{I}$. The unimodularity condition reads

$$f(x,y,z) \equiv \det\mathbb{S} = x^3 + y^3 + z^3 - 3xyz = 1 \tag{8}$$

and inscribes in xyz-space a cubic surface that resembles a witch's hat (is sometimes called "Jonas' hexenhut") with axis coincident with the principal diagonal of the unit cube. The set of such matrices is multiplicatively closed

$$\{x_1,y_1,z_1\} * \{x_2,y_2,z_2\} = \{x_{12},y_{12},z_{12}\} \tag{9}$$
$$\begin{aligned} x_{12} &= x_1x_2 + y_1z_2 + z_1y_2 \\ y_{12} &= x_1y_2 + y_1x_2 + z_1z_2 \\ z_{12} &= x_1z_2 + y_1y_2 + z_1x_2 \end{aligned}$$

Associativity is automatic, commutativity is established by calculation, and the identity lives obviously at $\{1,0,0\}$. Finally, matrix inversion supplies

$$\{x,y,z\}_{\text{inverse}} = \{x^2 - yz, z^2 - xy, y^2 - zx\}$$

The set of $\mathbb{S}$-matrices constitutes therefore a representation of a certain Abelian group.

From product of determinants = determinant of product we have the remarkable identity

$$f(x_1,y_1,z_1)f(x_2,y_2,z_2) = f(x_{12},y_{12},z_{12}) \tag{10}$$

which can be looked upon as a generalization of Brahmagupta's identity, the 3^{rd}-order member of an infinite tower of such identities of ascending order. From (10) and the structure (8) of f we see that if (x_1,y_1,z_1) are integers that satisfy $f(x_1,y_1,z_1) = 1$, and if so also are (x_2,y_2,z_2), then so also are the integers (x_{12},y_{12},z_{12}). In particular, if $f(x,y,z) = 1$ then so does

$$f(x^2 + 2yz, z^2 + 2xy, y^2 + 2zx) = 1 \tag{11}$$

Sebbar does not attempt to exhibit integer solutions of $f(x,y,z) = 1$. Trivial solutions are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. But when introduced into (10) or (11) those simply cycle among themselves. We confront therefore the

PROBLEM: Exhibit a non-trivial integral solution of

$$x^3 + y^3 + z^3 - 3xyz = 1$$

Some ramifications. Returning briefly—for motivational purposes—to the Pell equation, the eigenvalues of the circulant matrix $\mathbb{M}$ are

$$\begin{aligned} u &= x + \sqrt{n}\,y \\ v &= x - \sqrt{n}\,y \end{aligned} \tag{12}$$

so we have

$$\det \mathbb{M} = uv$$

Look to the differential operator

$$\Box \equiv \partial_x^2 - \tfrac{1}{n}\partial_y^2 = \begin{cases} \text{Laplacian in the case } n = -1 \\ \text{d'Alembertian in the case } n = +1 \end{cases}$$

Passing by means of (12) from xy-variables to uv-variables, we by

$$\partial_x = \partial_u + \partial_v, \quad \partial_y = \sqrt{n}\,(\partial_u - \partial_v)$$

have

$$\Box = 4\,\partial_u\partial_v$$

Solutions of $\Box W(u,v) = 0$ are of the form[2]

$$W(u,v) = F(u) + G(v) = F(x + \sqrt{n}\,y) + G(x - \sqrt{n}\,y)$$

of which a particular (the "fundamental") instance is

$$\mathcal{W}(u,v) = \log uv = \log(x^2 - ny^2)$$

This function is real/complex according as $x^2 - ny^2 \gtrless 0$ (it is, in particular, real on the Pell-Fermat conic $x^2 - ny^2 = 1$) and singular on the lines asymptotic to the conic: $u = v = 0$. By translational displacement we obtain the solution

$$\mathcal{W}_h(x,y) = \log(x^2 - ny^2 - 2hx + h^2)$$

which on the conic becomes $w_h(x) = \log(1 - 2hx + h^2)$. This function is proportional to a generator—familiar from 2-dimensional potential theory—of the Chebyschev polynomials of the first kind:

$$-\tfrac{1}{2}w_h(x) = \log\frac{1}{\sqrt{1 - 2hx + h^2}} = \sum_{k=1}^{\infty} T_k(x)\,\frac{h^k}{k}$$

The handbooks remind us that those polynomials satisfy the recursion relation

$$T_{k+1}(x) = 2x\,T_k(x) - T_{k-1}(x)$$

the differential equations

$$(1 - x^2)T_k{}''(x) - x\,T_k{}'(x) + k^2 T_k(x) = 0$$

and the orthonormality relations

$$\int_{-1}^{+1} T_j(x)T_k(x)\frac{dx}{\sqrt{1 - x^2}} = \begin{cases} 0 & : \quad j \neq k \\ \pi & : \quad j = k = 0 \\ \tfrac{1}{2}\pi & : \quad j = k \neq 0 \end{cases}$$

[2] This result is most familiar as encountered in 1-dimensional wave theory.

For discussion of another connection between the Pell equation and Chebyshev polynomials of the 1^{st} and 2^{nd} kinds, see the Wikipedia articles "Pell's Equation" (cited previously) and "Chebyshev Polynomials."

It is within the context of those remarks that (with Sebbar) we observe that the eigenvalues of 3-dimensional circulant matrix $\mathbb{S}$ are the complex numbers u, v, w given by[3]

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where

$$\begin{aligned} \omega &= -\tfrac{1}{2} + i\,\tfrac{1}{2}\sqrt{3} = (1)^{\frac{1}{3}} \\ \omega^2 &= -\tfrac{1}{2} - i\,\tfrac{1}{2}\sqrt{3} \end{aligned}$$

Inversely

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tfrac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

So

$$\begin{aligned} \partial_u &= \tfrac{1}{3}(\partial_x + \partial_y + \partial_z) \\ \partial_v &= \tfrac{1}{3}(\partial_x + \omega^2\partial_y + \omega\partial_z) \\ \partial_w &= \tfrac{1}{3}(\partial_x + \omega\partial_y + \omega^2\partial_z) \end{aligned}$$

from which we obtain Sebbar's 3^{rd}-order differential operator

$$\Box \equiv 27\,\partial_u\partial_v\partial_w = \partial_x^3 + \partial_y^3 + \partial_z^3 - 3\,\partial_x\partial_y\partial_z$$

We have

$$\Box\, W(u,v,w) = 0 \quad \Longleftrightarrow \quad W(u,v,w) = F(u,v) + G(u,w) + H(v,w)$$

or which a particular instance is

$$\mathcal{W}(u,v,w) = \log(u\,v\,w) = \log(x^3 + y^3 + x^3 - 3xyz)$$

We have

$$\det \mathbb{S} = u\,v\,w = x^3 + y^3 + x^3 - 3xyz$$

so the singularity of $\mathcal{W}(u,v,w)$ coincides with the singularity of $\mathbb{S}$.

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[3] These results are characteristic of the general theory of circulant matrices; see the Wikepedia article "Circulant Matrices." In the 2-dimensional theory we might, in this spirit, have written

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \omega \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with} \quad \omega = -1 = (1)^{\frac{1}{2}}$$

Translational displacement produces the solution

$$\begin{aligned}\mathcal{W}_h(u,v,w) &= \log[(x-h)^3+y^3+x^3-3(x-h)yz]\\ &= \log[(x^3+y^3+z^3-3xyz)+3xh^2-h^3-3h(x^2-yz)]\end{aligned}$$

which on the circular curve where the cubic surface $x^2-yz=0$ intersects the hexenhut $x^3+y^3+z^3-3xyz=1$ becomes

$$w_h(x)=\log(1+3xh^2-h^3)=\sum_{n=0}^{\infty}h^nP_n(x)$$

with

$$\begin{aligned}
P_0 &= 0 & P_9 &= -\tfrac{1}{3}+27x^3\\
P_1 &= 0 & P_{10} &= -\tfrac{27}{2}x^2+\tfrac{243}{5}x^5\\
P_2 &= 3x & P_{11} &= 3x-81x^4\\
P_3 &= -1 & P_{12} &= -\tfrac{1}{4}+54x^3-\tfrac{243}{2}x^6\\
P_4 &= -\tfrac{9}{2}x^2 & P_{13} &= -18x^2+243x^5\\
P_5 &= 3x & P_{14} &= 3x-\tfrac{405}{2}x^4+\tfrac{2187}{7}x^7\\
P_6 &= -\tfrac{1}{2}+9x^3 & P_{15} &= -\tfrac{1}{5}+90x^3-729x^6\\
P_7 &= -9x^2 & P_{16} &= -\tfrac{45}{2}x^2+729x^5-\tfrac{6561}{8}x^8\\
P_8 &= 3x-\tfrac{81}{4}x^4 & P_{17} &= 3x-405x^4+2187x^7
\end{aligned}$$

Sebbar asserts that "these polynomials are very interesting; they satisfy a four-term recursion relation, a third-order differential equation, and are related to elliptic functions." To me they seem, however, quite *un*interesting; $P_n(x)$ is not of order n, and they are not linearly independent—we have, for example, the identities

$$\begin{aligned}
P_0 &= P_1\\
P_2 &= P_5\\
P_4 &= \tfrac{1}{2}P_7\\
P_9 &= 3P_6-\tfrac{7}{6}P_3\\
P_{13} &= 5P_{10}-\tfrac{11}{2}P_7\\
P_{15} &= 6P_{12}-\tfrac{26}{3}P_9+\tfrac{143}{90}P_3\\
P_{17} &= 7P_{14}-\tfrac{25}{2}P_{11}+\tfrac{39}{6}P_5\\
&\ \vdots
\end{aligned}$$

so orthogonality is out of question.

Nonexistence of Pell triples. Returning to the PROBLEM posed on page 3: It is clear that if one of the variables $\{x, y, z\}$ in

$$x^3 + y^3 + z^3 - 3xyz = 1$$

is 0 then the others must be $\{1, 0\}$, so we are led back to the three trivial cases mentioned on page 3. It is clear also that if $\{x, y, z\}$ are some permutation of

{even, even, even} then $x^3 + y^3 + z^3$ is even and $3xyz$ is even
{even, even, odd } then $x^3 + y^3 + z^3$ is odd and $3xyz$ is even
{even, odd, odd } then $x^3 + y^3 + z^3$ is even and $3xyz$ is even
{ odd, odd, odd } then $x^3 + y^3 + z^3$ is odd and $3xyz$ is odd

so their difference can be 1 (odd) only in cases of the type {even, even, odd}. In the least such case we have

$$2^3 + 2^3 + 1^3 - 3 \cdot 2 \cdot 2 \cdot 1 = 5$$

I conclude that "Pell triples" do not exist, and suspect that it was because Sebbar was aware of this elementary fact that he does not even mention the integer aspects of his "generalized Pell problem."